Affine Lie algebra via Lyndon words

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Basic definitions

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1. $[a, b]$ is an alternating ($[a, b] = -[b, a]$) bilinear map
2. $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$

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- *Ideal* of a Lie algebra $\mathfrak{g}$ is a subspace I such that $[\mathfrak{g}, I] \subseteq I$.
- Lie algebra is called *Simple* if it has no proper nonzero ideals.

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Weight space $V_\psi^{\mathfrak{h}} = \{v | \pi(h)(v) = \psi(h)v \text{ for all } h \in \mathfrak{h}\}$

$\psi\colon \mathfrak{h} \to \mathbb{R}$ $\qquad$ $\pi\colon \mathfrak{h} \to End(V)$

Basic definitions

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Reduced root system is a set (E, Δ), where E is a finite-dimensional Euclidean space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(\cdot, \cdot)$ and Δ is a finite subset, such that:

- $0 \notin \Delta; \mathbb{R}\Delta = V$;
- If $\alpha \in \Delta$, then $n\alpha \in \Delta$ if and only if $n = \pm 1$;
- For $\alpha, \beta \in \Delta$, the projection of β onto α is in $\{0, \pm\frac{\alpha}{2}, \pm\alpha\}$
- If $\alpha, \beta \in \Delta$, then reflection $s_\alpha(\beta) \in \Delta$, where

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We call a root system *indecomposable* if it cannot be expressed as $A \cup B$ for some sets A and B such that $\forall a \in A, b \in B$ we have $(a, b) = 0$.

Elements of a root system are called *roots*.

Basic definitions

- Set of *positive* roots Δ^+ is subset of Δ, such that it doesn't contain $-\alpha$ and α simultaneously and for any two distinct $\alpha, \beta \in \Delta^+$ such that $\alpha + \beta \in \Delta$, we have $\alpha + \beta \in \Delta^+$. Such a set is not unique.

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- Root is called *simple* if it cannot be written as a sum of two elements of Δ^+.

- Let us consider an indecomposable root system of finite type:

$$\Delta^+ \sqcup \Delta^- \subset Q$$

(where Q denotes the root lattice) associated with the symmetric pairing:

$$(\cdot,\cdot)\colon Q \otimes Q \to \mathbb{Z}$$

- Let $\{\alpha_i\}_{i \in I}$ denote a choice of simple roots.

Cartan matrix

- The Cartan matrix $(a_{ij})_{i,j\in I}$ and the symmetrized Cartan matrix $(d_{ij})_{i,j\in I}$ of this root system are:

$$a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

and

$$d_{ij} = (\alpha_i, \alpha_j)$$

- It is well-known that the following Lie algebra associated with an indecomposable root system Δ is **simple**. All simple finite-dimensional Lie algebras arise that way.

Definition

$$\mathfrak{g} = \mathbb{Q}\langle e_i, f_i, h_i \rangle_{i \in I} \Big/ \text{relations 1 - 3}$$

where we impose the following relations for all $i, j \in I$:

1. $\underbrace{[e_i, [e_i, \ldots [e_i, [e_i, e_j]] \ldots]]}_{1-a_{ij} \ \text{Lie brackets}} = 0, \quad \text{if } i \neq j$
2. $[h_j, e_i] = d_{ji} e_i, \quad [h_j, h_i] = 0$
3. $[e_i, f_j] = \delta_i^j h_i$

as well as the opposite relations with e's replaced by f's.

- Such Lie algebra has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^{+} \oplus \mathfrak{h} \oplus \mathfrak{n}^{-}$$

where $\mathfrak{n}^{+}$, $\mathfrak{h}$, $\mathfrak{n}^{-}$ are the Lie subalgebras of $\mathfrak{g}$ generated by the e_i, h_i, f_i, respectively.

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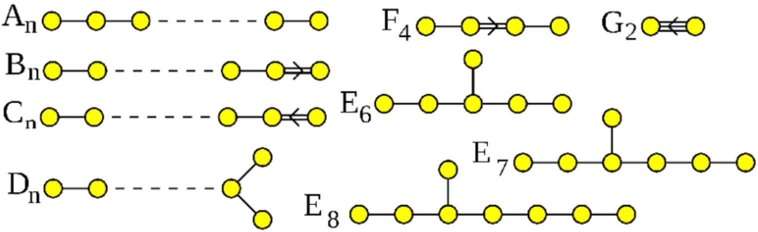
- It is well-known that we can decompose $\mathfrak{n}^+$ as follows:

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathbb{Q} \cdot e_\alpha$$

- Each Cartan matrix can be illustrated by a *Dynkin diagram*.
- If a root system is indecomposable, then the corresponding Dynkin diagram looks like one of the following diagrams:



- Let us introduce *Affine Lie algebra* with a trivial central charge (a.k.a. loop Lie algebra)

$$L\mathfrak{g} = \mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{Q}[t, t^{-1}]$$

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- The triangular decomposition extends to a similar decomposition:

$$L\mathfrak{g} = L\mathfrak{n}^+ \oplus L\mathfrak{h} \oplus L\mathfrak{n}^-$$

- We think of $L\mathfrak{n}^+$ as being generated by:

$$e_i^{(d)} = e_i \otimes t^d \qquad \forall\, i \in I, d \in \mathbb{Z}.$$

- Associate to $e_i^{(d)}$ the *letter* $i^{(d)}$; call d the *exponent* of $i^{(d)}$.
- The letters $\{i^{(d)}\}_{i \in I}^{d \in \mathbb{Z}}$ form our *Alphabet*.
- Any word in our alphabet will be called a *loop word*:

$$\left[i_1^{(d_1)} \dots i_k^{(d_k)}\right]$$

- We fix a set of weights $C = \{c_i\}_{i \in I}$ with $c_i \in \mathbb{Z}_{>0}$ for all i
- Let us fix an order on I.
- For the rest of the presentation, we fix the following order on our alphabet for $L\mathfrak{n}^+$:

$$i^{(d)} < j^{(e)} \iff \begin{cases} \frac{d}{c_i} > \frac{e}{c_j} \\ or \\ \frac{d}{c_i} = \frac{e}{c_j} \text{ and } i < j \end{cases}$$

- Now this induces lexicographic order on a set of loop words.

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- Any Lyndon word ℓ has a *costandard* factorization: $\ell = \ell_1 \ell_2$ such that ℓ_2 is the longest proper suffix of ℓ which is Lyndon, in which case ℓ_1 turns out to be Lyndon as well.
- For any Lyndon word ℓ, we define $e_\ell \in L\mathfrak{n}^+$ inductively by $e_{[i^{(d)}]} = e_i^{(d)}$ for $i \in I$ and $d \in \mathbb{Z}$ and:

$$e_\ell = [e_{\ell_1}, e_{\ell_2}] \in L\mathfrak{n}^+,$$

where $\ell = \ell_1 \ell_2$ is the above costandard factorization.

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- Since we have an infinite alphabet $\{i^{(d)}\}_{i\in I}^{d\in\mathbb{Z}}$, we want to extend this definition to our affine case.

Filtration

- The following filtration is a slight generalization of the approach of Neguț-Tsymbaliuk.

$$L\mathfrak{n}^+ = \bigcup_{s=0}^{\infty} L^{(s)}\mathfrak{n}^+$$

defined for the finite-dimensional Lie subalgebras:

$$L^{(s)}\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \bigoplus_{-s\cdot f(\alpha)\leq d\leq s\cdot f(\alpha)} \mathbb{Q}\cdot e_\alpha^{(d)} \subset L\mathfrak{n}^+$$

where $e_\alpha^{(d)} = e_\alpha \otimes t^d$ and $f(\alpha)$ denotes the *weighted height*:

$$f(\alpha) = \sum_{i\in I} k_i \cdot c_i \quad \text{if} \quad \alpha = \sum_{i\in I} k_i \cdot \alpha_i$$

- We can apply the definition of standard Lyndon word to each $L^{(s)}\mathfrak{n}^+$, and we want to show that it does not depend on s.

Leclerc rule

Theorem

There exists a bijection:

$$\ell\colon \left\{(\alpha, d) \in \Delta^+ \times \mathbb{Z} \,\middle|\, |d| \leq s \cdot f(\alpha)\right\} \xrightarrow{\sim} \left\{\textit{standard Lyndon loop words for } L^{(s)}\mathfrak{n}^+\right\}$$

explicitly determined by $\ell(\alpha_i, d) = [i^{(d)}]$ *and Leclerc rule:*

$$\ell(\alpha, d) = \max_{\substack{(\gamma_1,d_1)+(\gamma_2,d_2)=(\alpha,d) \\ \gamma_k \in \Delta^+,\ |d_k| \leq s f(\gamma_k) \\ \ell(\gamma_1,d_1) < \ell(\gamma_2,d_2)}} \left\{\textit{concatenation } \ell(\gamma_1, d_1)\ell(\gamma_2, d_2)\right\}$$

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Periodicity, Convexity and Monotonicity properties

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$$\ell(\alpha, d) = \left[i_1^{(d_1)} \dots i_k^{(d_k)} \right] \Longrightarrow$$

$$\ell(\alpha, d + f(\alpha)) = \left[i_1^{(d_1 + c_1)} \dots i_k^{(d_k + c_k)} \right]$$

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Example for $\mathfrak{sl}_4$, $c_1 = c_2 = 3, c_3 = 5$, and order $1 < 2 < 3$:
$\ell(\alpha_1 + \alpha_2 + \alpha_3, 5) = [3^{(3)}2^{(1)}1^{(1)}]$
$\ell(\alpha_1 + \alpha_2 + \alpha_3, 16) = [3^{(8)}2^{(4)}1^{(4)}]$

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$$\ell(\alpha, d) < \ell(\alpha + \beta, d + t) < \ell(\beta, t)$$

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- *Monotonicity:*

$$\ell(\alpha, d + 1) < \ell(\alpha, d) \qquad \forall\, (\alpha, d) \in \Delta^+ \times \mathbb{Z}$$

The Exponent Rule

- A word $w = \left[i_1^{(d_1)} \dots i_n^{(d_n)} \right]$ is called *exponent-tight* if

$$i_k^{(d_k)} \geq i_r^{(d_r+1)} \qquad \text{for all} \quad 1 \leq k, r \leq n.$$

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- **Corollary**: $\ell(\alpha, d)$ is a permutation of letters of the maximal Lyndon word of the given degree (α, d).

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